

NORM RESOLVENT CONVERGENCE OF SINGULARLY SCALED SCHRÖDINGER OPERATORS AND δ' -POTENTIALS

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ABSTRACT. For a real-valued function V from the Faddeev–Marchenko class, we prove the norm resolvent convergence, as $\varepsilon \rightarrow 0$, of a family S_ε of one-dimensional Schrödinger operators on the line of the form

$$S_\varepsilon := -\frac{d^2}{dx^2} + \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right).$$

Under certain conditions the functions $\varepsilon^{-2}V(x/\varepsilon)$ converge in the sense of distributions as $\varepsilon \rightarrow 0$ to $\delta'(x)$, and then the limit S_0 of S_ε might be considered as a “physically motivated” interpretation of the one-dimensional Schrödinger operator with potential δ' .

1. INTRODUCTION

The aim of this paper is to study convergence as $\varepsilon \rightarrow 0$ of the family S_ε of Schrödinger operators on the line given by

$$S_\varepsilon := -\frac{d^2}{dx^2} + \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right), \quad (1.1)$$

for the largest possible class of potentials V , namely, for real-valued V in the Faddeev–Marchenko class $L_1(\mathbb{R}; (1 + |x|) dx)$. The asymptotic behaviour of S_ε and their many-dimensional analogues has been discussed in both the mathematical and physical literature in connection with the small-energy scattering [2, 28] and singular perturbations [5, 44, 45] since 1980-ies. Recently, the Schrödinger operator family S_ε has been enjoying a renewed interest motivated by the question on approximation of thin quantum waveguides by quantum graphs [1, 13–15, 19, 22].

In three dimensions, the corresponding family of Hamiltonians

$$H_\varepsilon := -\Delta + \frac{\lambda(\varepsilon)}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right)$$

was first studied by Albeverio and Høegh-Krohn [5]. Here $\lambda(\varepsilon)$ is a smooth function with $\lambda(0) = 1$ and $\lambda'(0) \neq 0$ and V is short-range and of Rollnik class [43, Ch. X.2]. It was shown that the family H_ε converges as $\varepsilon \rightarrow 0$ in the strong resolvent sense to H_0 that is either the free Hamiltonian $-\Delta$ or its perturbation by a delta-function (see the pioneering work by Berezin and Faddeev [8]) depending on whether or not there is a zero-energy resonance for $H = -\Delta + V$. We recall that H is said to possess a zero-energy resonance if the equation $H\psi = 0$ has a distributional solution that is bounded but does not belong to $L_2(\mathbb{R}^3)$. Analogous results were established in [5] also for V containing finitely or infinitely many summands scaled about different centres.

In [2], the authors discussed the low-energy scattering in two particle non-relativistic quantum mechanics. They used the results of [5] and the connection between the low-energy behaviour of the scattering amplitude and scattering matrix for the corresponding Hamiltonian $H = -\Delta + V$ in $L_2(\mathbb{R}^3)$ and for the scaled Hamiltonians $H_\varepsilon = -\Delta + \varepsilon^{-2}V(x/\varepsilon)$ as $\varepsilon \rightarrow 0$ to study in detail possible resonant and non-resonant cases. Similar problem for Hamiltonians including the Coulomb-type interaction was treated in [4].

Interestingly enough, the low-energy scattering theory for Schrödinger operators in dimensions one and two is more complicated than in dimension three. This is connected with, respectively, the square root and logarithmic singularities the Green function of the free Hamiltonian then possesses. The low-energy scattering for the one-dimensional Schrödinger operator S_1 and its connection to the behaviour of S_ε as $\varepsilon \rightarrow 0$ was thoroughly investigated by Bollé, Gesztesy, Klaus, and Wilk, both for the non-resonant [11] and resonant [10] cases respectively; in dimension two, the low-energy asymptotics was discussed in [9]. Continuity of the scattering matrix at zero energy for one-dimensional Schrödinger operators with Faddeev–Marchenko potentials in the resonant case was independently established by Guseinov in [25] and by Klaus in [32].

Another reason to study the family S_ε comes from the quantum graph theory. One of the fundamental questions of this theory consists of justifying the possibility of approximating dynamics of a quantum particle confined to real-world mesoscopic waveguides of small width ε by its dynamics on the idealized one-dimensional “manifolds” obtained in the limit as ε vanishes.

For instance, for bent waveguides Ω_ε in dimension 2 that coincide with two straight strips outside a compact “vertex” region this question was studied in [1]. The authors showed that the problem reduces to establishing the norm resolvent convergence of the operator family S_ε of (1.1) as $\varepsilon \rightarrow 0$. Assuming that V has nonzero mean and decays exponentially, i.e., that

$$\int_{\mathbb{R}} V(x) dx \neq 0 \quad \text{and} \quad e^{a|\cdot|} V \in L_1(\mathbb{R}) \quad \text{for some } a > 0, \quad (1.2)$$

norm resolvent convergence of S_ε as $\varepsilon \rightarrow 0$ was proved and the limiting operator S_0 was identified. Depending on whether or not the potential V is resonant, the operator S_0 is either the direct sum $S_- \oplus S_+$ of two free half-line Schrödinger operators given by $S_\pm := -\frac{d^2}{dx^2}$ on \mathbb{R}_\pm and subject to the Dirichlet condition $y(0) = 0$ at the origin, or a singular perturbation $S(\theta)$ of the free Schrödinger operator S defined by

$$S(\theta)y = -y'' \quad (1.3)$$

on functions y in $W_2^2(\mathbb{R} \setminus \{0\})$ obeying the nontrivial interface conditions at the origin,

$$y(0+) = \theta y(0-), \quad \theta y'(0+) = y'(0-). \quad (1.4)$$

The number θ depends on the geometric properties (in particular, on the curvature) of the bent waveguide. The convergence results for bent waveguides were recently re-examined from a different viewpoint in the paper [15].

In [13], the analysis of [1] was further extended to the case of waveguides Ω_ε with non-trivial scaling properties around the vertex region; namely, the authors demonstrated that in the resonant case the class of limiting Hamiltonians is wider and includes, in particular, perturbations of the free Schrödinger operator S by the delta-functions. Results analogous to those of [1] and [13] were also established in [14] in the case of Robin conditions on the boundary of Ω_ε .

The effect of twisting in 3D quantum wave-guides with shrinking non-circular cross-section was investigated in [22]. The problem was reduced to the study of convergence of singularly scaled Schrödinger-type operators \mathcal{H}_ε in an unbounded tube with potentials containing the singular term $\varepsilon^{-2}\vartheta(x_1/\varepsilon)$, where x_1 is a longitudinal coordinate in the tube. The function ϑ of compact support describes the “fast” twisting of the cross-section over a small x_1 -interval shrinking to a point x_0 . It was proved that the family \mathcal{H}_ε converges as $\varepsilon \rightarrow 0$ in the (suitably understood) norm resolvent sense to the operator \mathcal{H}_0 , which is a one-dimensional Schrödinger operator subject to the Dirichlet boundary condition at x_0 .

Earlier, the operators \mathcal{H}_ε appeared in [35] in connection with the problem of large-time behaviour of solutions to the heat equation in a twisted tube, and their strong resolvent convergence was established therein.

The limiting behaviour of the family S_ε is interesting for yet another reason. Namely, in the case where V has a zero mean and its first moment is -1 , i.e., where

$$\int_{\mathbb{R}} V(x) dx = 0, \quad \int_{\mathbb{R}} xV(x) dx = -1,$$

the family of scaled potentials $V_\varepsilon(x) := \varepsilon^{-2}V(\varepsilon^{-1}x)$ converges as $\varepsilon \rightarrow 0$ to the derivative δ' of the Dirac delta-function. Therefore, the limit S_0 of S_ε can be regarded as a physically motivated realization of the free Hamiltonian S perturbed by the *singular potential* δ' . We notice that there is a related but completely different notion of δ' -interaction; namely, following a widely accepted agreement, the Hamiltonian $S_{\beta, \delta'}$ with δ' -interaction that is given formally by

$$-\frac{d^2}{dx^2} - \beta \langle \cdot, \delta' \rangle \delta',$$

should be interpreted as the free Schrödinger operator $S_{\beta, \delta'} y = -y''$ acting on the domain

$$\begin{aligned} \text{dom } S_{\beta, \delta'} := \{y \in W_2^2(\mathbb{R} \setminus \{0\}) \mid y'(0-) = y'(0+) =: y'(0), \\ y(0+) - y(0-) = \beta y'(0)\}, \end{aligned}$$

see [3, 6]. However, there is no clear physical motivation for this particular choice and, moreover, $S_{\beta, \delta'}$ cannot be used for defining a free Hamiltonian with δ' -potential [42]. We note that δ' -interactions and, more generally, singular point interactions for Schrödinger operators in dimension one and higher have widely been discussed in both the mathematical and physical literature; see [7, 12, 20, 27, 34, 36, 41, 46] and also the extensive bibliography lists in the monographs [3, 6, 33].

Šeba [44] was seemingly the first to realize the connection between the limiting behaviour of S_ε and δ' -perturbations of the Schrödinger operators. The paper [44] discussed an even wider class of problems including S_ε as a particular case; however, its results for the family (1.1) erroneously state that the only possible norm-resolvent limit of S_ε is the direct sum $S_- \oplus S_+$ of the free half-line Schrödinger operators subject to the Dirichlet boundary condition at $x = 0$. Such a result would suggest that in dimension 1 no non-trivial definition of the Schrödinger operator with potential δ' is possible. Recalling that the Schrödinger operators are quantum mechanical Hamiltonians for a particle on the line, one would have to conclude that, in dimension 1, the δ' potential barrier is completely opaque, i.e., that the particle cannot tunnel through it.

However, such a conclusion is in contradiction with the numerical analysis of exactly solvable models of (1.1) with piece-wise constant V of compact support performed recently by Zolotaryuk a.o. [17], [48–51]. Namely, the authors demonstrated that for resonant V , the limiting value of the transmission coefficient $T_\varepsilon(k)$ of the operator S_ε is different from zero, thus indicating that the limiting operator S_0 cannot be given by $S_- \oplus S_+$. In certain cases, S_0 was identified with the operator $S(\theta)$ of (1.3), (1.4). The operator of the form $S(\theta)$ also appear in [38, 42] as a realization of the pseudo-Hamiltonian $-\frac{d^2}{dx^2} + \alpha \delta'(x)$ by means of the distribution theory over discontinuous test functions. Yet another evidence that the convergence result of [44] cannot be true was derived in the paper [24], where eigenvalue and eigenfunction asymptotics as $\varepsilon \rightarrow 0$ were studied for the full-line Schrödinger operators given by the differential expression

$$-\frac{d^2}{dx^2} + \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right) + W(x),$$

where V is regular and of compact support and W is unbounded at infinity. Namely, the eigenfunctions were shown to satisfy in the limit the Dirichlet condition $y(0) = 0$ in the non-resonant case and the interface condition (1.4) in a resonant case, thus again exhibiting the zero-energy dichotomy.

The above results motivated us to re-examine the convergence of the Schrödinger operator family (1.1) for real-valued V not necessarily satisfying (1.2). In our previous paper [23], we treated the case where V is of compact support, taken $[-1, 1]$ for definiteness, and, in particular, confirmed the result established in [1, 13, 14] under the restriction (1.2). In [23, 24], a real-valued potential V was called *resonant* if the Neumann Sturm–Liouville operator \mathcal{N} given by

$$\mathcal{N}y := -y'' + Vy$$

on functions in the Sobolev space $W_2^2(-1, 1)$ satisfying the conditions $y'(-1) = 0$, $y'(1) = 0$ has a non-trivial null-space. This definition of a resonant case agrees with the one taken in [1]. We proved in [23] that S_ε converge as $\varepsilon \rightarrow 0$ in the norm resolvent sense to the limit operator S_0 that is equal to $S_- \oplus S_+$ in the non-resonance case and to $S(\theta)$ of (1.3), (1.4) in the resonance case with $\theta := u(1)/u(-1)$, where u is an eigenfunction of \mathcal{N} corresponding to the eigenvalue zero. The eigenfunction is unique up to a constant factor, so that the number θ is well defined.

The main goal of this paper is to extend the convergence result of [23] to the set of real-valued potentials V of the Faddeev–Marchenko class. In particular, we do not assume that V is of non-zero mean or that V decays exponentially. Nevertheless, we shall prove that the results on convergence of S_ε established in [1, 13, 14, 23] for particular cases remain valid.

Recall [31] that the Schrödinger operator $S_1 = -\frac{d^2}{dx^2} + V$ is said to possess a *zero-energy resonance* (or *half-bound state*) if there exists a solution y to the equation

$$-y'' + Vy = 0$$

that is bounded on the whole line; the corresponding potential V is then called *resonant*. Such a solution y is then unique up to a scalar factor and has nonzero limits $y(\pm\infty)$ at $\pm\infty$, so that the number

$$\theta := \frac{y(+\infty)}{y(-\infty)} \tag{1.5}$$

is well defined. Our main result reads as follows.

Theorem 1.1. *Assume that V is real valued and belongs to the Faddeev–Marchenko class. Then the operator family (1.1) converges as $\varepsilon \rightarrow 0$ in the norm resolvent sense, and the limit S_0 is equal to the direct sum $S_- \oplus S_+$ of the Dirichlet half-line Schrödinger operators S_\pm in the non-resonant case, and to the operator $S(\theta)$ defined by (1.3), (1.4), with θ of (1.5), in the resonant case.*

We establish Theorem 1.1 in two steps. On the first step, we adapt our approach of [23] and establish the theorem with the potential $V_\varepsilon := \varepsilon^{-2}V(\varepsilon^{-1} \cdot)$ truncated onto the contracting intervals $(-x_\varepsilon, x_\varepsilon)$, for a suitably defined x_ε . On the second step, we show that the part of the potential V_ε outside $(-x_\varepsilon, x_\varepsilon)$ introduces a perturbation that is too weak to affect the limit of the resolvents as $\varepsilon \rightarrow 0$. In both the resonant and non-resonant cases, the reasoning is based on a careful asymptotic analysis of the Jost solutions for S_ε and S_1 at infinity and in the vicinity of the splitting points $\pm x_\varepsilon$.

It should be noted that the above theorem seemingly cannot be extended beyond the Faddeev–Marchenko class of potentials V . Indeed, take $V(x) = (1+x^2)^{-1}$; then the family $V_\varepsilon(x) := (\varepsilon^2 + x^2)^{-1}$ increases as ε decreases to zero. The quadratic forms \mathfrak{s}_ε generated

by the Schrödinger operators S_ε are closed and well defined on $W_2^1(\mathbb{R})$. For $y \in W_2^1(\mathbb{R})$ the limit

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0+} \mathfrak{s}_\varepsilon(y, y) &= \int_{\mathbb{R}} |y'(x)|^2 dx + \lim_{\varepsilon \rightarrow 0+} \int_{\mathbb{R}} \frac{|y(x)|^2 dx}{\varepsilon^2 + x^2} \\ &= \int_{\mathbb{R}} |y'(x)|^2 dx + \int_{\mathbb{R}} \frac{|y(x)|^2 dx}{x^2} =: \mathfrak{s}_0(y, y) \end{aligned}$$

is finite if and only if $y(0) = 0$; the “only if” part is clear, and the “if” part follows from the fact that the Hardy operator,

$$g \mapsto \frac{1}{x} \int_0^x g(t) dt$$

is bounded in $L_2(0, 1)$ [26, Ch. 9.9]. By Theorem S.14 of [43], the quadratic form \mathfrak{s}_0 is closed on the domain $\{y \in W_2^1(\mathbb{R}) \mid y(0) = 0\}$ and the Schrödinger operators S_ε converge in the strong resolvent sense as $\varepsilon \rightarrow 0+$ to the operator S_0 corresponding to the quadratic form \mathfrak{s}_0 . Since $\text{dom } S_0 \subset \text{dom } \mathfrak{s}_0$ by the first representation theorem [29, Theorem VI.2.1], we conclude that S_0 is the direct sum of two half-line Schrödinger operators S_- and S_+ with the Bessel potential $1/x^2$. Thus there is no resonance effect present here and the limiting operator contains a nontrivial potential, in contrast to the situation of Theorem 1.1.

It is worth noting that the operators S_ε given by (1.1) are scale-invariant and therefore it should be expected that the limit operator S_0 possesses the same property. And indeed, both in the non-resonant and resonant cases the limit operators $S_- \oplus S_+$ and $S(\theta)$ describe the so-called scale-invariant point interactions [21].

We also remark that one can construct infinitely many resonant potentials starting with an arbitrary compactly supported $V \in L_1(\mathbb{R})$ and considering a family $\{\alpha V\}$ with a real coupling constant α . As shown in [23] (see also [17], [47]–[51] for various step-like potentials V), there is an infinite discrete set Σ of α ’s such that αV is resonant for every $\alpha \in \Sigma$. The same conclusion holds if V is supported by the whole line but decays rapidly at infinity. We are going to discuss this question elsewhere.

The rest of the paper is organized as follows. In Section 2, we recall the definition and some important properties of the Jost solutions of the Schrödinger equation; also, the low-energy and large- x asymptotics of the Jost solutions and of some special solutions are established therein. In Section 3, we study the norm resolvent convergence of the auxiliary family of Schrödinger operators \tilde{S}_ε with potentials $\chi_\varepsilon V_\varepsilon$, where χ_ε is the characteristic function of a neighbourhood of the origin squeezing to zero as $\varepsilon \rightarrow 0$. Section 4 is devoted to investigation of the resolvent of \tilde{S}_ε via the subtle asymptotic analysis of the Jost solutions of this operator. Simultaneously, we describe the limit behaviour of scattering coefficients for the operator \tilde{S}_ε . In these three sections, the non-resonant and resonant cases have to be treated separately. Finally, in the last section we establish proximity of the operator families S_ε and \tilde{S}_ε in the norm resolvent sense, which, in combination with Theorem 3.2, gives a complete proof of the main result.

Notations. Throughout the paper, C_1, C_2, C_3, C_4 , and C_5 shall stand for the constants of Proposition 2.1 and Lemma 2.5. Letters c_j denote various positive numbers independent of ε , whose values might be different in different proofs, and $\|f\|$ stands for the $L_2(\mathbb{R})$ -norm of a function f .

2. ASYMPTOTICS OF JOST SOLUTIONS

2.1. Large- x behaviour at low energies. Throughout the paper V denotes a fixed real-valued potential of the Faddeev–Marchenko class, i.e., satisfying

$$\int_{\mathbb{R}} (1 + |t|) |V(t)| dt < \infty.$$

Next, we denote by $f_+(\cdot, k)$ and $f_-(\cdot, k)$ respectively the *right* and *left Jost solutions* of the Schrödinger equation

$$-y'' + Vy = k^2 y \quad (2.1)$$

for complex k with $\operatorname{Im} k \geq 0$. The Jost solution $f_+(x, k)$ is asymptotic to e^{ikx} as $x \rightarrow +\infty$, while $f_-(x, k)$ is asymptotic to e^{-ikx} as $x \rightarrow -\infty$. For non-zero k , the Jost solutions exist whenever the potential V is integrable at $\pm\infty$ [16, Ch. I.1.3] or even under somewhat weaker assumptions [30]; however, for V of the Faddeev–Marchenko class they have some special properties. Namely, set

$$\begin{aligned} \sigma_-(x) &:= \int_{-\infty}^x |V(t)| dt, & \tau_-(x) &:= \int_{-\infty}^x (1 + |t|) |V(t)| dt, \\ \sigma_+(x) &:= \int_x^{\infty} |V(t)| dt, & \tau_+(x) &:= \int_x^{\infty} (1 + |t|) |V(t)| dt; \end{aligned}$$

then the claims of [18, Lemma 1] and [40, Lemma 3.1.3] can be stated as follows.

Proposition 2.1. *Assume that the potential V is of the Faddeev–Marchenko class. Then there are numbers C_j , $j = 1, \dots, 4$, such that the following holds for every k in the closed upper-half complex plane:*

$$|f_+(x, k) - e^{ikx}| \leq C_1 |e^{ikx}| \tau_+(x), \quad x \geq 0; \quad (2.2)$$

$$|f_+(x, k) - e^{ikx}| \leq C_2 |e^{ikx}| (1 + |x|), \quad x < 0, \quad (2.3)$$

$$|f'_+(x, k) - ik f_+(x, k)| \leq C_3 |e^{ikx}| \sigma_+(x), \quad x \geq 0, \quad (2.4)$$

$$|f'_+(x, k) - ik f_+(x, k)| \leq C_4 |e^{ikx}|, \quad x < 0. \quad (2.5)$$

Similar estimates hold for $f_-(\cdot, k)$.

In what follows, we shall consider restrictions of V_ε onto contracting intervals $(x_\varepsilon, x_\varepsilon)$, the choice of x_ε being tailored for the given V in a special way. We start with the following observation.

Lemma 2.2. *For V of the Faddeev–Marchenko class there exists an even, continuous, and positive function $\rho_V: \mathbb{R} \rightarrow \mathbb{R}$ such that*

- (i) ρ_V strictly increases for $x > 0$;
- (ii) $|x|^{-1} \rho_V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$;
- (iii) $\int_{\mathbb{R}} \rho_V(x) |V(x)| dx < \infty$.

Proof. For V of compact support, set $\rho_V(x) := 1 + x^2$. Otherwise the function

$$\tau(x) := \tau_+(|x|) + \tau_-(-|x|) = \int_{|t| > |x|} (1 + |t|) |V(t)| dt$$

is strictly positive and even; moreover, it does not increase for $x > 0$ and vanishes at infinity. We set

$$\rho_V(x) := \frac{1 + |x|}{\tau^\alpha(x)}$$

for $\alpha \in (0, 1)$; then (i) and (ii) are immediate, while (iii) follows from

$$\begin{aligned} \int_{\mathbb{R}} \rho_V(x) |V(x)| dx &= \int_0^\infty \frac{(1+x)(|V(x)| + |V(-x)|)}{\tau^\alpha(x)} dx \\ &= - \int_0^\infty \frac{\tau'(x)}{\tau^\alpha(x)} dx = \frac{\tau^{1-\alpha}(0)}{1-\alpha} < \infty. \end{aligned}$$

The proof is complete. \square

Fix ρ_V as in Lemma 2.2 and for $\varepsilon > 0$ denote by ξ_ε the unique positive solution of the equation $\rho_V(\xi) = 1/\varepsilon$. Such a solution ξ_ε exists for all sufficiently small positive ε . It follows from (i) and (ii) of Lemma 2.2 that $\xi_\varepsilon \rightarrow +\infty$ and $x_\varepsilon := \varepsilon \xi_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. This choice of x_ε and ξ_ε is fixed for the rest of the paper.

Set $W\{f, g\} := fg' - f'g$ to be the Wronskian of functions f and g and define $D(k)$ as the Wronskian of the Jost solutions $f_+(\cdot, k)$ and $f_-(\cdot, k)$,

$$D(k) := W\{f_+(\cdot, k), f_-(\cdot, k)\} = f_+(x, k)f'_-(x, k) - f'_+(x, k)f_-(x, k).$$

Fix a nonzero $k \in \overline{\mathbb{C}^+}$; then Proposition 2.1 implies the following asymptotic behaviour of the Jost solutions.

Lemma 2.3. *The following holds as $\varepsilon \rightarrow 0$:*

$$f_+(\xi_\varepsilon, \varepsilon k) \rightarrow 1, \quad f_-(-\xi_\varepsilon, \varepsilon k) \rightarrow 1; \quad (2.6)$$

$$\varepsilon f_+(-\xi_\varepsilon, \varepsilon k) \rightarrow 0, \quad \varepsilon f_-(-\xi_\varepsilon, \varepsilon k) \rightarrow 0; \quad (2.7)$$

$$\varepsilon^{-1} f'_+(\xi_\varepsilon, \varepsilon k) \rightarrow ik, \quad \varepsilon^{-1} f'_-(-\xi_\varepsilon, \varepsilon k) \rightarrow -ik; \quad (2.8)$$

$$f'_+(-\xi_\varepsilon, \varepsilon k) \rightarrow -D(0), \quad f'_-(\xi_\varepsilon, \varepsilon k) \rightarrow D(0). \quad (2.9)$$

Proof. Relations (2.6) follow immediately from (2.2) and an analogous estimate for f_- . Also, (2.3) and a similar inequality for f_- imply that

$$|f_+(-\xi_\varepsilon, \varepsilon k)| + |f_-(-\xi_\varepsilon, \varepsilon k)| \leq c(1 + \xi_\varepsilon)$$

for some $c > 0$ independent of ε ; since $\varepsilon \xi_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, formulae (2.7) follow.

Next, estimate (2.4) yields

$$|\varepsilon^{-1} f'_+(\xi_\varepsilon, \varepsilon k) - ik| \leq C_3 \varepsilon^{-1} \sigma_+(\xi_\varepsilon) + |k| |f_+(\xi_\varepsilon, \varepsilon k) - 1|. \quad (2.10)$$

Now the choice of the ξ_ε and the properties of the function ρ_V justified in Lemma 2.2 show that

$$\begin{aligned} \varepsilon^{-1} \sigma_+(\xi_\varepsilon) &= \varepsilon^{-1} \int_{\xi_\varepsilon}^\infty |V(t)| dt \\ &\leq \varepsilon^{-1} \int_{\xi_\varepsilon}^\infty \frac{\rho_V(t) |V(t)|}{\rho_V(\xi_\varepsilon)} dt = \int_{\xi_\varepsilon}^\infty \rho_V(t) |V(t)| dt \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$. A passage to the limit in (2.10) establishes the first relation of (2.8); the second one is justified similarly.

Finally, combination of the equalities

$$D(\varepsilon k) = \begin{vmatrix} \varepsilon f_+(-\xi_\varepsilon, \varepsilon k) & f_-(-\xi_\varepsilon, \varepsilon k) \\ f'_+(-\xi_\varepsilon, \varepsilon k) & \varepsilon^{-1} f'_-(-\xi_\varepsilon, \varepsilon k) \end{vmatrix} = \begin{vmatrix} f_+(\xi_\varepsilon, \varepsilon k) & \varepsilon f_-(-\xi_\varepsilon, \varepsilon k) \\ \varepsilon^{-1} f'_+(\xi_\varepsilon, \varepsilon k) & f'_-(-\xi_\varepsilon, \varepsilon k) \end{vmatrix}$$

with relations (2.6)–(2.8) proved above results in (2.9). The proof is complete. \square

2.2. Refinement in the resonant case. In the resonant case, the Jost solutions f_+ and f_- become linearly dependent at $k = 0$, i.e.,

$$f_-(\cdot, 0) = \theta f_+(\cdot, 0) \quad (2.11)$$

for some real nonzero θ . The corresponding half-bound state y can be given as $y = c_- f_-(\cdot, 0)$ or $y = c_+ f_+(\cdot, 0)$, with some constants c_\pm satisfying the relation $c_+ = \theta c_-$. Since $y(\pm\infty) = c_\pm$ by the definition of the Jost solutions, the number θ as defined by (2.11) is the same as in (1.5).

Moreover, in this case, more precise information on the asymptotic behaviour of $f_\pm(\cdot, 0)$ at infinity is available. Namely, we shall prove that the solutions $f_\pm(\cdot, \varepsilon k)$ remain then bounded over $(-\xi_\varepsilon, \xi_\varepsilon)$ uniformly in ε and shall re-examine the convergence in (2.7) and (2.9). To do this, we need another pair of solutions to equation (2.1).

Lemma 2.4. *There exist two solutions $g_+(\cdot, k)$ and $g_-(\cdot, k)$ of (2.1) satisfying the relations $W\{f_+, g_+\} = 1$ and $W\{f_-, g_-\} = -1$ and such that, for some $x_+ > 0$, $x_- < 0$ and all $k \in \mathbb{C}^+$, the following inequalities hold:*

$$|g_+(x, k)| \leq 6|e^{-ikx}| x, \quad x > x_+, \quad (2.12)$$

$$|g_-(x, k)| \leq 6|e^{ikx}| |x|, \quad x < x_-. \quad (2.13)$$

Proof. It follows from the estimate (2.2) that there is an $x_+ > 0$ such that $f_+(x, k) \neq 0$ for every $k \in \overline{\mathbb{C}^+}$ and every $x \geq x_+$. Then the function

$$g_+(x, k) := f_+(x, k) \int_{x_+}^x \frac{dt}{f_+^2(t, k)}$$

is well defined for $x \geq x_+$, solves there equation (2.1), and verifies the relation $W\{f_+, g_+\} = 1$. Clearly, g_+ can uniquely be continued to the whole axis as a solution of (2.1).

Without loss of generality we can (and shall) assume that x_+ is so large that $C_1 \tau_+(x) \leq \frac{1}{2}$ for all $k \in \overline{\mathbb{C}^+}$ and all $x > x_+$, C_1 being the constant of (2.2). Then

$$\frac{1}{2}|e^{ikx}| \leq |f_+(x, k)| \leq \frac{3}{2}|e^{ikx}|$$

for such k and x ; therefore,

$$\frac{|e^{ikx} g_+(x, k)|}{x} \leq \frac{6|e^{2ikx}|}{x - x_+} \int_{x_+}^x \frac{dt}{|e^{2ikt}|} \leq 6,$$

thus yielding (2.12).

By similar arguments, there exists an $x_- < 0$ and a solution $g_-(\cdot, k)$ of (2.1) that for all $x < x_-$ is equal to

$$g_-(x, k) := f_-(x, k) \int_x^{x_-} \frac{dt}{f_-^2(t, k)}$$

and satisfies the relation $W\{f_-, g_-\} = -1$ and the inequality (2.13). \square

Lemma 2.5. *Assume that V is resonant and define θ via (2.11). Then for some $C_5 > 0$ and all ε small enough we have*

$$\max_{|x| \leq \xi_\varepsilon} |f_\pm(x, \varepsilon k)| \leq C_5. \quad (2.14)$$

Furthermore, the following holds as $\varepsilon \rightarrow 0$:

$$f_+(-\xi_\varepsilon, \varepsilon k) \rightarrow \theta^{-1}, \quad f_-(\xi_\varepsilon, \varepsilon k) \rightarrow \theta, \quad (2.15)$$

$$\varepsilon^{-1} f'_+(-\xi_\varepsilon, \varepsilon k) \rightarrow ik\theta, \quad \varepsilon^{-1} f'_-(\xi_\varepsilon, \varepsilon k) \rightarrow -ik\theta^{-1}. \quad (2.16)$$

Proof. We shall only prove (2.14) for the solution f_+ , since the proof for f_- is completely analogous. Clearly, there are α_ε and β_ε such that

$$f_+(\cdot, \varepsilon k) = \alpha_\varepsilon f_-(\cdot, \varepsilon k) + \beta_\varepsilon g_-(\cdot, \varepsilon k),$$

and it is easy to see that $\beta_\varepsilon = D(\varepsilon k)$. Recall that $D(0) = 0$ in the resonant case; moreover, equation (2.29) of [32] implies that D is differentiable at $k = 0$ and that the derivative $\dot{D}(0)$ is

$$\dot{D}(0) = -i(\theta + \theta^{-1}) \neq 0.$$

It follows that

$$\lim_{\varepsilon \rightarrow 0} \frac{D(\varepsilon k)}{\varepsilon} = -ik(\theta + \theta^{-1}); \quad (2.17)$$

thus $\beta_\varepsilon = O(\varepsilon)$ as $\varepsilon \rightarrow 0$, and in combination with (2.13) this shows that

$$\max_{x \in [-\xi_\varepsilon, x_-]} |f_+(x, \varepsilon k) - \alpha_\varepsilon f_-(x, \varepsilon k)| \leq 6|\beta_\varepsilon| e^{|x_\varepsilon k|} \xi_\varepsilon = O(x_\varepsilon) \quad (2.18)$$

as $\varepsilon \rightarrow 0$. For each fixed x , $f_\pm(x, k)$ are continuous functions of $k \in \overline{\mathbb{C}^+}$. We conclude from this and the relation (2.18) that $\alpha_\varepsilon \rightarrow \theta^{-1}$ as $\varepsilon \rightarrow 0$. Therefore there exists $c > 0$ such that, for all ε small enough,

$$\max_{x \in [-\xi_\varepsilon, x_-]} |f_+(x, \varepsilon k)| \leq 2\theta^{-1} \max_{x \in [-\xi_\varepsilon, x_-]} |f_-(x, \varepsilon k)| + 1 \leq c.$$

A similar bound over $x_- \leq x \leq \xi_+$ follows from (2.2) and (2.3), thus establishing (2.14) for f_+ .

Specifying now (2.18) to $x = -\xi_\varepsilon$ and recalling (2.6), we get

$$\lim_{\varepsilon \rightarrow 0} f_+(-\xi_\varepsilon, \varepsilon k) = \theta^{-1}.$$

Observing that

$$\begin{aligned} f_+(-\xi_\varepsilon, \varepsilon k) \frac{f'_-(-\xi_\varepsilon, \varepsilon k)}{\varepsilon} - \frac{f'_+(-\xi_\varepsilon, \varepsilon k)}{\varepsilon} f_-(-\xi_\varepsilon, \varepsilon k) \\ = \frac{D(\varepsilon k)}{\varepsilon} \rightarrow -ik(\theta + \theta^{-1}) \end{aligned}$$

as $\varepsilon \rightarrow 0$ by (2.17), we deduce from Lemma 2.3 and (2.15) that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} f'_+(-\xi_\varepsilon, \varepsilon k) = ik\theta.$$

The behaviour of the Jost solution f_- on the right-half line can be analyzed similarly. \square

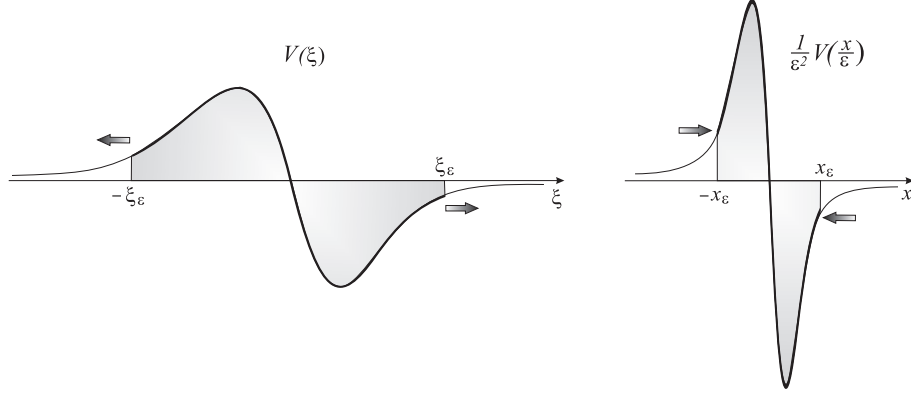
3. CONVERGENCE OF THE AUXILIARY OPERATOR FAMILY \tilde{S}_ε

3.1. A general approach. In what follows, we denote by χ_ε the characteristic function of the interval $[-x_\varepsilon, x_\varepsilon]$ and consider an auxiliary family \tilde{S}_ε of Schrödinger operators on the line defined via

$$\tilde{S}_\varepsilon := -\frac{d^2}{dx^2} + \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right) \chi_\varepsilon(x).$$

Based on the results of the papers [1, 13, 23, 24] we expect that the family \tilde{S}_ε converges to a limit S_0 as $\varepsilon \rightarrow 0$ in the norm resolvent sense and that S_0 is $S_- \oplus S_+$ in the non-resonant case and $S(\theta)$, with θ of (1.5) or (2.11), in the resonant case.

To explain the main idea behind the constructions that follow, we fix a nonreal k^2 , take an arbitrary $f \in L_2(\mathbb{R})$, and set $y_0 := (S_0 - k^2)^{-1}f$ and $\tilde{y}_\varepsilon := (\tilde{S}_\varepsilon - k^2)^{-1}f$. To show that $\tilde{y}_\varepsilon \rightarrow y_0$ as $\varepsilon \rightarrow 0$, we construct an auxiliary function y_ε in $\text{dom } \tilde{S}_\varepsilon$ such that

FIGURE 1. Potentials in ξ and x coordinates

$q_\varepsilon := y_\varepsilon - y_0 \rightarrow 0$ and $r_\varepsilon := (\tilde{S}_\varepsilon - k^2)y_\varepsilon - f \rightarrow 0$ in the L_2 -norm as $\varepsilon \rightarrow 0$; it then follows that

$$\tilde{y}_\varepsilon - y_0 = (y_\varepsilon - y_0) - (y_\varepsilon - \tilde{y}_\varepsilon) = q_\varepsilon - (\tilde{S}_\varepsilon - k^2)^{-1}r_\varepsilon \rightarrow 0.$$

A more careful analysis shows that the convergence of \tilde{y}_ε to y_0 is uniform in f on bounded sets, thus establishing the norm resolvent convergence of \tilde{S}_ε to S_0 .

We start the construction of the approximation y_ε by observing that the function $\tilde{y}_\varepsilon = (\tilde{S}_\varepsilon - k^2)^{-1}f$ solves the equations

$$-y'' = k^2y + f$$

for $|x| > x_\varepsilon$ and

$$-y'' + \varepsilon^{-2}V(x/\varepsilon)y = k^2y + f$$

for $|x| < x_\varepsilon$. Setting $\tilde{y}_\varepsilon(x) := w(x/\varepsilon)$ in the “fast variable” domain $|x| < x_\varepsilon$, we see that w must solve the equation

$$-w''(\xi) + V(\xi)w(\xi) = \varepsilon^2k^2w(\xi) + \varepsilon^2f(\varepsilon\xi)$$

for $\xi \in (-\xi_\varepsilon, \xi_\varepsilon)$, see Figure 1. A general solution to this equation has the form $w_\varepsilon + u_\varepsilon$, where

$$w_\varepsilon = a_-(\varepsilon)f_-(\cdot, \varepsilon k) + a_+(\varepsilon)f_+(\cdot, \varepsilon k),$$

with $a_\pm(\varepsilon)$ arbitrary complex numbers, is a solution to the homogeneous problem and

$$\begin{aligned} u_\varepsilon(\xi) := & \frac{\varepsilon^2 f_+(\xi, \varepsilon k)}{D(\varepsilon k)} \int_{-\xi_\varepsilon}^{\xi} f_-(\eta, \varepsilon k) f(\varepsilon \eta) d\eta \\ & + \frac{\varepsilon^2 f_-(\xi, \varepsilon k)}{D(\varepsilon k)} \int_{\xi}^{\xi_\varepsilon} f_+(\eta, \varepsilon k) f(\varepsilon \eta) d\eta \end{aligned}$$

is a particular solution.

It is therefore natural to take a function

$$z_\varepsilon(x) := y_0(x)[1 - \chi_\varepsilon(x)] + [w_\varepsilon(x/\varepsilon) + u_\varepsilon(x/\varepsilon)]\chi_\varepsilon(x)$$

equal to y_0 for $|x| > x_\varepsilon$ and to $w_\varepsilon(x/\varepsilon) + u_\varepsilon(x/\varepsilon)$ for $|x| < x_\varepsilon$ as a leading term of the asymptotic expansion of y_ε as $\varepsilon \rightarrow 0$. The function z_ε belongs to W_2^2 outside the points $x = \pm x_\varepsilon$ and has jump discontinuities at these points; namely, denoting by $[g]_a$ the jump of a function g at a point $x = a$, we see that

$$\begin{aligned} |z_\varepsilon|_{\pm x_\varepsilon} &= |y_0(\pm x_\varepsilon) - w_\varepsilon(\pm \xi_\varepsilon) - u_\varepsilon(\pm \xi_\varepsilon)|, \\ |z'_\varepsilon|_{\pm x_\varepsilon} &= |y'_0(\pm x_\varepsilon) - \varepsilon^{-1}w'_\varepsilon(\pm \xi_\varepsilon) - \varepsilon^{-1}u'_\varepsilon(\pm \xi_\varepsilon)|. \end{aligned} \tag{3.1}$$

Our task is to choose the coefficients $a_+(\varepsilon)$ and $a_-(\varepsilon)$ defining w_ε in such a way that the values of the right hand sides of (3.1) vanish as $\varepsilon \rightarrow 0$. We first show that the particular solution u_ε does not contribute significantly to these values. Set

$$\Delta_\varepsilon := [-\xi_\varepsilon, \xi_\varepsilon], \quad \Delta_\varepsilon^- := [-\xi_\varepsilon, 0], \quad \Delta_\varepsilon^+ := [0, \xi_\varepsilon].$$

Lemma 3.1. *The following holds as $\varepsilon \rightarrow 0$:*

$$|u_\varepsilon(\pm\xi_\varepsilon)| = O(x_\varepsilon^{1/2})\|f\|, \quad (3.2)$$

$$\varepsilon^{-1}|u'_\varepsilon(\pm\xi_\varepsilon)| = O(x_\varepsilon^{1/2})\|f\|, \quad (3.3)$$

and, in addition, $\|u_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\| = O(x_\varepsilon)\|f\|$.

Proof. First, observe that

$$\begin{aligned} \max_{\Delta_\varepsilon^+} |u_\varepsilon| &\leq \frac{2\varepsilon^2}{|D(\varepsilon k)|} \max_{\Delta_\varepsilon^+} |f_+(\cdot, \varepsilon k)| \max_{\Delta_\varepsilon} |f_-(\cdot, \varepsilon k)| \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon\eta)| \, d\eta, \\ \max_{\Delta_\varepsilon^-} |u_\varepsilon| &\leq \frac{2\varepsilon^2}{|D(\varepsilon k)|} \max_{\Delta_\varepsilon} |f_+(\cdot, \varepsilon k)| \max_{\Delta_\varepsilon^-} |f_-(\cdot, \varepsilon k)| \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon\eta)| \, d\eta \end{aligned}$$

and that

$$\int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon\eta)| \, d\eta \leq \sqrt{2} \varepsilon^{-1/2} \xi_\varepsilon^{1/2} \|f\| \quad (3.4)$$

by the Cauchy–Bunyakovski inequality.

In the non-resonant case, $D(0) \neq 0$, and by virtue of Proposition 2.1 we have

$$\max_{\Delta_\varepsilon} |u_\varepsilon| \leq c_1 \varepsilon^{3/2} (1 + \xi_\varepsilon) \xi_\varepsilon^{1/2} \|f\| \leq 2c_1 x_\varepsilon^{3/2} \|f\|$$

for all sufficiently small ε and some $c_1 > 0$ independent of ε . In the resonant case, we use (2.17) and the fact that, by Lemma 2.5, the Jost solutions $f_\pm(\cdot, \varepsilon k)$ remain bounded over Δ_ε uniformly in ε to find c_2 independent of ε such that

$$\max_{\Delta_\varepsilon} |u_\varepsilon| \leq c_2 x_\varepsilon^{1/2} \|f\|.$$

Since the last inequality holds also in the non-resonant case for small enough ε , we readily deduce (3.2) and estimate the L_2 -norm of $u_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon$ via

$$\int_{-x_\varepsilon}^{x_\varepsilon} |u_\varepsilon(x/\varepsilon)|^2 \, dx = \varepsilon \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |u_\varepsilon(\xi)|^2 \, d\xi \leq 2c_2^2 \varepsilon x_\varepsilon \xi_\varepsilon \|f\|^2 = O(x_\varepsilon^2) \|f\|^2$$

as $\varepsilon \rightarrow 0$.

Next we compute the derivative

$$u'_\varepsilon(\xi_\varepsilon) = \frac{\varepsilon^2 f'_+(\xi_\varepsilon, \varepsilon k)}{D(\varepsilon k)} \int_{-\xi_\varepsilon}^{\xi_\varepsilon} f_-(\eta, \varepsilon k) f(\varepsilon\eta) \, d\eta$$

and obtain

$$\begin{aligned} \varepsilon^{-1}|u'_\varepsilon(\xi_\varepsilon)| &\leq \frac{\varepsilon}{|D(\varepsilon k)|} |f'_+(\xi_\varepsilon, \varepsilon k)| \max_{\Delta_\varepsilon} |f_-(\xi, \varepsilon k)| \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon\eta)| \, d\eta \\ &\leq \frac{c_3 |k| \varepsilon x_\varepsilon^{1/2}}{|D(\varepsilon k)|} \max_{\Delta_\varepsilon} |f_-(\xi, \varepsilon k)| \|f\| \end{aligned}$$

due to (2.8) and (3.4). Applying Proposition 2.1 in the non-resonant case and Lemma 2.5 along with (2.17) in the resonant one, we find that

$$\varepsilon^{-1}|u'_\varepsilon(\xi_\varepsilon^{1/2})| \leq c_4 x_\varepsilon \|f\|.$$

Similar considerations yield the estimate for $u'_\varepsilon(-\xi_\varepsilon)$. \square

Next we need to choose the coefficients $a_\pm(\varepsilon)$ in a proper way. This will be done separately in the non-resonant and resonant cases.

3.2. A special solution in the non-resonant case. Observing that $y_0(\pm x_\varepsilon) \rightarrow 0$ and that $y'_0(\pm x_\varepsilon) \rightarrow y'_0(0\pm)$ as $\varepsilon \rightarrow 0$ and recalling that the values $|f'_\pm(\mp \xi_\varepsilon, \varepsilon k)|$ converge to the positive number $|D(0)|$ by (2.9), one concludes that a suitable choice of $a_\pm(\varepsilon)$ might be

$$a_-(\varepsilon) := \varepsilon y'_0(x_\varepsilon)/f'_-(\xi_\varepsilon, \varepsilon k), \quad a_+(\varepsilon) := \varepsilon y'_0(-x_\varepsilon)/f'_+(-\xi_\varepsilon, \varepsilon k).$$

And indeed, the relations of Proposition 2.1 and Lemma 2.3 yield then the asymptotics

$$\begin{aligned} w_\varepsilon(\pm \xi_\varepsilon) &= \frac{\varepsilon f_-(\pm \xi_\varepsilon, \varepsilon k)}{f'_-(\xi_\varepsilon, \varepsilon k)} y'_0(x_\varepsilon) + \frac{\varepsilon f_+(\pm \xi_\varepsilon, \varepsilon k)}{f'_+(-\xi_\varepsilon, \varepsilon k)} y'_0(-x_\varepsilon) \\ &= O(x_\varepsilon) \cdot [y'_0(x_\varepsilon) + y'_0(-x_\varepsilon)], \\ \varepsilon^{-1} w'_\varepsilon(\pm \xi_\varepsilon) &= \frac{f'_-(\pm \xi_\varepsilon, \varepsilon k)}{f'_-(\xi_\varepsilon, \varepsilon k)} y'_0(x_\varepsilon) + \frac{f'_+(\pm \xi_\varepsilon, \varepsilon k)}{f'_+(-\xi_\varepsilon, \varepsilon k)} y'_0(-x_\varepsilon) \\ &= y'_0(\pm x_\varepsilon) + O(\varepsilon) \cdot y'_0(\mp x_\varepsilon). \end{aligned}$$

Since the resolvent of S_0 acts continuously from $L_2(\mathbb{R})$ into $W_2^2(\mathbb{R} \setminus \{0\})$, there exists a constant c_1 independent of f such that

$$\|y_0\|_{W_2^2(\mathbb{R} \setminus \{0\})} \leq c_1 \|f\|;$$

thus the Sobolev embedding theorem yields the estimates

$$\sup_x (|y_0(x)| + |y'_0(x)|) \leq c_2 \|y_0\|_{W_2^2(\mathbb{R} \setminus \{0\})} \leq c_3 \|f\| \quad (3.5)$$

as well as the relations

$$\begin{aligned} |y_0(\pm x_\varepsilon)| &= \left| \int_0^{\pm x_\varepsilon} y'_0(t) dt \right| \leq x_\varepsilon^{1/2} \|y'_0\| \leq c_4 x_\varepsilon^{1/2} \|f\|, \\ |y'_0(\pm x_\varepsilon) - y'_0(0\pm)| &= \left| \int_0^{\pm x_\varepsilon} y''_0(t) dt \right| \leq x_\varepsilon^{1/2} \|y''_0\| \leq c_5 x_\varepsilon^{1/2} \|f\|. \end{aligned}$$

Returning now to formulae (3.1) and combining the above estimates with Lemma 3.1, we get

$$\begin{aligned} |[z_\varepsilon]_{\pm x_\varepsilon}| &\leq |y_0(\pm x_\varepsilon)| + |w_\varepsilon(\pm \xi_\varepsilon)| + |u_\varepsilon(\pm \xi_\varepsilon)| = O(x_\varepsilon^{1/2}) \cdot \|f\|, \\ |[z'_\varepsilon]_{\pm x_\varepsilon}| &\leq |y'_0(\pm x_\varepsilon) - \varepsilon^{-1} w'_\varepsilon(\pm \xi_\varepsilon)| + \varepsilon^{-1} |u'_\varepsilon(\pm \xi_\varepsilon)| = O(x_\varepsilon^{1/2}) \cdot \|f\|. \end{aligned}$$

We shall also need an estimate on the L_2 -norm of $w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon$. Since $a_\pm(\varepsilon) = O(\varepsilon)\|f\|$ and

$$\int_{-x_\varepsilon}^{x_\varepsilon} |f_\pm(x/\varepsilon, \varepsilon k)|^2 dx = \varepsilon \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f_\pm(\xi, \varepsilon k)|^2 d\xi \leq c_6 \varepsilon (1 + \xi_\varepsilon)^3,$$

we find that, as $\varepsilon \rightarrow 0$,

$$\|w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\| = O(x_\varepsilon^{3/2}) \|f\|.$$

3.3. A special solution in the resonant case. In the resonant case, the limit operator S_0 is expected to be $S(\theta)$, with θ of (2.11). The main difference with the non-resonant case is that the zero energy Jost solutions f_- and f_+ are now linearly dependent and thus follow quite a different asymptotics.

A possible choice for $a_{\pm}(\varepsilon)$ could be the one resulting in the equalities

$$w_{\varepsilon}(-\xi_{\varepsilon}) = y_0(-x_{\varepsilon}), \quad \varepsilon^{-1}w'_{\varepsilon}(-\xi_{\varepsilon}) = y'_0(-x_{\varepsilon}).$$

However, we have found it more convenient to take the limiting values $a_{\pm}(0)$ instead of such $a_{\pm}(\varepsilon)$ for all $\varepsilon > 0$, i.e., to set

$$\begin{aligned} a_{-}(\varepsilon) &\equiv \frac{-1}{ik(\theta + \theta^{-1})} [y'_0(0-)\theta^{-1} - ik\theta y_0(0-)], \\ a_{+}(\varepsilon) &\equiv \frac{1}{ik(\theta + \theta^{-1})} [iky_0(0-) + y'_0(0-)]. \end{aligned}$$

Then, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} w_{\varepsilon}(-\xi_{\varepsilon}) &= y_0(0-) \frac{\theta f_{-}(-\xi_{\varepsilon}, \varepsilon k) + f_{+}(-\xi_{\varepsilon}, \varepsilon k)}{\theta + \theta^{-1}} \\ &\quad + y'_0(0-) \frac{-\theta^{-1} f_{-}(-\xi_{\varepsilon}, \varepsilon k) + f_{+}(-\xi_{\varepsilon}, \varepsilon k)}{ik(\theta + \theta^{-1})} \\ &= y_0(0-)(1 + o(1)) + y'_0(0-) \cdot o(1) \\ &= y_0(-x_{\varepsilon}) + o(1)\|f\| \end{aligned}$$

and, similarly,

$$\begin{aligned} \varepsilon^{-1}w'_{\varepsilon}(-\xi_{\varepsilon}) &= y_0(0-) \frac{\theta f'_{-}(-\xi_{\varepsilon}, \varepsilon k) + f'_{+}(-\xi_{\varepsilon}, \varepsilon k)}{\varepsilon(\theta + \theta^{-1})} \\ &\quad + y'_0(0-) \frac{-\theta^{-1} f'_{-}(-\xi_{\varepsilon}, \varepsilon k) + f'_{+}(-\xi_{\varepsilon}, \varepsilon k)}{i\varepsilon k(\theta + \theta^{-1})} \\ &= y_0(0-) \cdot o(1) + y'_0(0-)(1 + o(1)) \\ &= y'_0(-x_{\varepsilon}) + o(1)\|f\|. \end{aligned}$$

Next, recalling the relations $y_0(0-) = \theta^{-1}y_0(0+)$ and $y'_0(0-) = \theta y'_0(0+)$, we see that

$$\begin{aligned} w_{\varepsilon}(\xi_{\varepsilon}) &= y_0(0+) \frac{f_{-}(\xi_{\varepsilon}, \varepsilon k) + \theta^{-1}f_{+}(\xi_{\varepsilon}, \varepsilon k)}{\theta + \theta^{-1}} \\ &\quad + y'_0(0+) \frac{-f_{-}(\xi_{\varepsilon}, \varepsilon k) + \theta f_{+}(\xi_{\varepsilon}, \varepsilon k)}{ik(\theta + \theta^{-1})} \\ &= y_0(0+)(1 + o(1)) + y'_0(0+) \cdot o(1) \\ &= y_0(x_{\varepsilon}) + o(1)\|f\| \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{-1}w'_{\varepsilon}(\xi_{\varepsilon}) &= y_0(0+) \frac{f'_{-}(\xi_{\varepsilon}, \varepsilon k) + \theta^{-1}f'_{+}(\xi_{\varepsilon}, \varepsilon k)}{\varepsilon(\theta + \theta^{-1})} \\ &\quad + y'_0(0+) \frac{-f'_{-}(\xi_{\varepsilon}, \varepsilon k) + \theta f'_{+}(\xi_{\varepsilon}, \varepsilon k)}{i\varepsilon k(\theta + \theta^{-1})} \\ &= y_0(0+) \cdot o(1) + y'_0(0+)(1 + o(1)) \\ &= y'_0(x_{\varepsilon}) + o(1) \cdot \|f\|. \end{aligned}$$

Combining the above relations, we conclude that the jumps of the function z_ε and its derivative at the points $\pm x_\varepsilon$ are small, namely that

$$\begin{aligned} [z_\varepsilon]_{\pm x_\varepsilon} &= o(1) \cdot \|f\|, \\ [z'_\varepsilon]_{\pm x_\varepsilon} &= o(1) \cdot \|f\| \end{aligned}$$

as $\varepsilon \rightarrow 0$.

Finally, using the fact that by Lemma 2.5 the Jost solutions $f_\pm(\cdot, \varepsilon k)$ remain uniformly bounded over $(-\xi_\varepsilon, \xi_\varepsilon)$ as $\varepsilon \rightarrow 0$, we arrive at the estimate

$$\int_{-x_\varepsilon}^{x_\varepsilon} |f_\pm(x/\varepsilon, \varepsilon k)|^2 dx = \varepsilon \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f_\pm(\xi, \varepsilon k)|^2 d\xi \leq c_7 \varepsilon \xi_\varepsilon = c_7 x_\varepsilon$$

for all small ε . Observing that $a_\pm(\varepsilon) = O(1)\|f\|$ as $\varepsilon \rightarrow 0$, we get the estimate

$$\|w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\| = O(x_\varepsilon) \|f\|.$$

3.4. Convergence of \tilde{S}_ε . We are now in a position to prove the main result of this section.

Theorem 3.2. *Denote by S_0 the operator $S_- \oplus S_+$ in the non-resonant case and the operator $S(\theta)$ with $\theta := f_-/f_+$ in the resonant case. Then the operator family \tilde{S}_ε converges in the norm resolvent sense as $\varepsilon \rightarrow 0$ to S_0 .*

Proof. Fix $k^2 \in \mathbb{C} \setminus \mathbb{R}$, take an arbitrary $f \in L_2(\mathbb{R})$, and set $y_0 := (S_0 - k^2)^{-1}f$. As the first step, we construct a function $q_\varepsilon \in L_2(\mathbb{R})$ with $\|q_\varepsilon\| = o(\|f\|)$ as $\varepsilon \rightarrow 0$ such that the function $y_\varepsilon := y_0 + q_\varepsilon$ belongs to $\text{dom } \tilde{S}_\varepsilon$ and, as $\varepsilon \rightarrow 0$, the relation

$$\|(\tilde{S}_\varepsilon - k^2)y_\varepsilon - f\| = o(\|f\|)$$

holds. We then show that this yields the required norm resolvent convergence.



FIGURE 2. Functions with the prescribed jumps at the origin

To construct such a y_ε , we shall modify z_ε at the points $\pm x_\varepsilon$ to eliminate the jumps. To this end let us introduce functions φ and ψ as in Figure 2 that are smooth outside the origin, have compact supports contained in $[0, \infty)$, and have the prescribed jumps $[\varphi]_0 = 1$, $[\varphi']_0 = 0$ and $[\psi]_0 = 0$, $[\psi']_0 = 1$. Set

$$\begin{aligned} \zeta_\varepsilon(x) &= [z_\varepsilon]_{-x_\varepsilon} \varphi(-x - x_\varepsilon) - [z'_\varepsilon]_{-x_\varepsilon} \psi(-x - x_\varepsilon) \\ &\quad - [z_\varepsilon]_{x_\varepsilon} \varphi(x - x_\varepsilon) - [z'_\varepsilon]_{x_\varepsilon} \psi(x - x_\varepsilon); \end{aligned}$$

then $\zeta_\varepsilon = 0$ on $(-x_\varepsilon, x_\varepsilon)$ and, in view of the estimates on $[z_\varepsilon]_{\pm x_\varepsilon}$ and $[z'_\varepsilon]_{\pm x_\varepsilon}$ of Subsections 3.2 and 3.3,

$$\max_{|x| > x_\varepsilon} |\zeta_\varepsilon^{(k)}(x)| = o(\|f\|) \quad (3.6)$$

as $\varepsilon \rightarrow 0$ for $k = 0, 1, 2$.

Clearly, the function $y_\varepsilon := z_\varepsilon + \zeta_\varepsilon$ is continuous on \mathbb{R} along with its derivative and belongs to $W_2^2(\mathbb{R}) = \text{dom } S_\varepsilon$. Observe that $y_\varepsilon = y_0$ for $|x|$ large enough; more exactly, we have

$$y_\varepsilon = y_0 + q_\varepsilon = (S_0 - k^2)^{-1}f + q_\varepsilon$$

with

$$q_\varepsilon := [w_\varepsilon(\cdot/\varepsilon) + u_\varepsilon(\cdot/\varepsilon) - y_0]\chi_\varepsilon + \zeta_\varepsilon$$

of compact support. Taking into account Lemma 3.1, the estimates on $\|w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\|$ derived above, and relations (3.5) and (3.6), we see that the L_2 -norm of the function q_ε is $o(\|f\|)$.

Next we calculate the function $(\tilde{S}_\varepsilon - k^2)y_\varepsilon$ for $|x| < x_\varepsilon$ and $|x| > x_\varepsilon$. If $|x| < x_\varepsilon$, then $\zeta_\varepsilon(x) = 0$, whence $y_\varepsilon(x) = z_\varepsilon(x)$ and

$$(\tilde{S}_\varepsilon - k^2)y_\varepsilon(x) = \varepsilon^{-2}[-z_\varepsilon''(\frac{x}{\varepsilon}) + V(\frac{x}{\varepsilon})z_\varepsilon(\frac{x}{\varepsilon})] - k^2z_\varepsilon(\frac{x}{\varepsilon}) = f(x).$$

If $|x| > x_\varepsilon$, then $y_\varepsilon = y_0 + \zeta_\varepsilon$, and hence

$$\begin{aligned} (\tilde{S}_\varepsilon - k^2)y_\varepsilon(x) &= \left(-\frac{d^2}{dx^2} - k^2\right)(y_0 + \zeta_\varepsilon)(x) \\ &= f(x) - \zeta_\varepsilon''(x) - k^2\zeta_\varepsilon(x). \end{aligned}$$

Therefore, $(\tilde{S}_\varepsilon - k^2)y_\varepsilon = f + r_\varepsilon$ with the remainder

$$r_\varepsilon(x) = -\zeta_\varepsilon''(x) - k^2\zeta_\varepsilon(x).$$

Using (3.6), we find that, as $\varepsilon \rightarrow 0$,

$$\|r_\varepsilon\| = o(\|f\|).$$

Finally, since $\|(\tilde{S}_\varepsilon - k^2)^{-1}\| \leq |\operatorname{Im} k^2|^{-1}$, it follows that

$$\begin{aligned} \|(\tilde{S}_\varepsilon - k^2)^{-1}f - (S_0 - k^2)^{-1}f\| &= \|\tilde{y}_\varepsilon - y_0\| \\ &\leq \|q_\varepsilon\| + \|(\tilde{S}_\varepsilon - k^2)^{-1}r_\varepsilon\| = o(\|f\|) \end{aligned}$$

as $\varepsilon \rightarrow 0$, and the proof is complete. \square

4. JOST SOLUTIONS AND SCATTERING COEFFICIENTS FOR THE OPERATOR \tilde{S}_ε

Our next task is to show that the resolvents of the operators \tilde{S}_ε and S_ε get closer as ε tends to zero. To do this, we shall need more information on the resolvent of the operator \tilde{S}_ε .

We denote by $\tilde{f}_\pm(\cdot, \varepsilon, k)$ the Jost solutions of the operator \tilde{S}_ε ; then the resolvent $\tilde{R}_\varepsilon(k) := (\tilde{S}_\varepsilon - k^2)^{-1}$ is an integral operator with kernel equal to the Green function $\tilde{G}_\varepsilon(x, y, k)$,

$$\tilde{G}_\varepsilon(x, y, k) = \frac{1}{\tilde{D}_\varepsilon(k)} \begin{cases} \tilde{f}_+(x, \varepsilon, k)\tilde{f}_-(y, \varepsilon, k), & x > y, \\ \tilde{f}_-(x, \varepsilon, k)\tilde{f}_+(y, \varepsilon, k), & x < y, \end{cases}$$

where $\tilde{D}_\varepsilon(k)$ is the Wronskian of the Jost solutions $\tilde{f}_+(\cdot, \varepsilon, k)$ and $\tilde{f}_-(\cdot, \varepsilon, k)$, i.e.,

$$\tilde{D}_\varepsilon(k) = \tilde{f}_+(\cdot, \varepsilon, k)\tilde{f}'_-(\cdot, \varepsilon, k) - \tilde{f}'_+(\cdot, \varepsilon, k)\tilde{f}_-(\cdot, \varepsilon, k).$$

In this section, we shall construct the Jost solutions $\tilde{f}_\pm(\cdot, \varepsilon, k)$ of the operator \tilde{S}_ε for a fixed $k \in \mathbb{C}$ with $\operatorname{Im} k > 0$ and $\operatorname{Re} k > 0$, and then study their behaviour as $\varepsilon \rightarrow 0$. The analysis of the right and left Jost solutions is quite similar, and only the case of \tilde{f}_+ will be investigated in detail. As a by-product, we establish some estimates on $\tilde{f}_\pm(\cdot, \varepsilon, k)$ that will essentially be used in the next section to prove the main result.

4.1. Jost solutions: a general construction. Since the potential of \tilde{S}_ε vanishes for $|x| > x_\varepsilon$, we consider separately three regions: $x > x_\varepsilon$, $|x| < x_\varepsilon$, and $x < -x_\varepsilon$.

Region 1: for $x > x_\varepsilon$ we clearly have

$$\tilde{f}_+(x, \varepsilon, k) = e^{ikx}.$$

Region 2: for $|x| < x_\varepsilon$ the Jost solution satisfies the equation

$$-y'' + \varepsilon^{-2}V(x/\varepsilon)y = k^2y$$

and thus is equal to

$$\tilde{f}_+(x, \varepsilon, k) = c_\varepsilon^+ f_+(x/\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(x/\varepsilon, \varepsilon k),$$

where f_\pm are the Jost solutions of (2.1), i.e., of the Schrödinger operator S_1 . Continuity of \tilde{f}_+ and its derivative at the point $x = x_\varepsilon$ results in the system

$$\begin{cases} c_\varepsilon^+ f_+(\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(\xi_\varepsilon, \varepsilon k) = e^{ikx_\varepsilon}, \\ c_\varepsilon^+ f'_+(\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f'_-(\xi_\varepsilon, \varepsilon k) = i\varepsilon k e^{ikx_\varepsilon}. \end{cases}$$

Recall that

$$D(k) := f_+(\xi, k)f'_-(\xi, k) - f'_+(\xi, k)f_-(\xi, k)$$

is the Wronskian of the Jost solutions f_+ and f_- and observe that $D(\varepsilon k) \neq 0$ for every positive ε since otherwise $f_+(\cdot, \varepsilon k)$ and $f_-(\cdot, \varepsilon k)$ would be linearly dependent and thus $\varepsilon^2 k^2 \in \mathbb{C}^+$ would be an eigenvalue of the self-adjoint operator S_1 . Solving the above system for c_ε^\pm , one gets

$$\begin{aligned} c_\varepsilon^+ &= \frac{e^{ikx_\varepsilon}}{D(\varepsilon k)} \left[f'_-(\xi_\varepsilon, \varepsilon k) - i\varepsilon k f_-(\xi_\varepsilon, \varepsilon k) \right]; \\ c_\varepsilon^- &= \frac{e^{ikx_\varepsilon}}{D(\varepsilon k)} \left[i\varepsilon k f_+(\xi_\varepsilon, \varepsilon k) - f'_+(\xi_\varepsilon, \varepsilon k) \right]. \end{aligned}$$

In the non-resonant case, $D(0) \neq 0$, and thus $|D(\varepsilon k)| \geq |D(0)|/2$ for all ε small enough by the continuity of D . Then $c_\varepsilon^- \rightarrow 0$ by (2.4), while $c_\varepsilon^+ \rightarrow 1$ by Lemma 2.3. In the resonant case, we combine this lemma with relation (2.17) and get the same convergence $c_\varepsilon^- \rightarrow 0$ and $c_\varepsilon^+ \rightarrow 1$ as $\varepsilon \rightarrow 0$.

Region 3: for $x < -x_\varepsilon$,

$$\tilde{f}_+(x, \varepsilon, k) = a_\varepsilon^+ e^{ikx} + b_\varepsilon^+ e^{-ikx}$$

for some coefficients a_ε^+ and b_ε^+ . Continuity of the Jost solution \tilde{f}_+ and its derivative at $x = -x_\varepsilon$ gives the linear system for a_ε^+ and b_ε^+ :

$$\begin{cases} a_\varepsilon^+ e^{-ikx_\varepsilon} + b_\varepsilon^+ e^{ikx_\varepsilon} = \tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k), \\ ik(a_\varepsilon^+ e^{-ikx_\varepsilon} - b_\varepsilon^+ e^{ikx_\varepsilon}) = \tilde{f}'_+(-x_\varepsilon + 0, \varepsilon, k), \end{cases}$$

solving which we find that

$$\begin{aligned} a_\varepsilon^+ &= \frac{e^{ikx_\varepsilon}}{2ik} \left[ik \tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) + \tilde{f}'_+(-x_\varepsilon + 0, \varepsilon, k) \right]; \\ b_\varepsilon^+ &= \frac{e^{-ikx_\varepsilon}}{2ik} \left[ik \tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) - \tilde{f}'_+(-x_\varepsilon + 0, \varepsilon, k) \right]. \end{aligned}$$

Remark 4.1. Calculating the Wronskian $\tilde{D}_\varepsilon(k)$ of the Jost solutions \tilde{f}_\pm at a point x to the left of $-x_\varepsilon$, one immediately sees that $\tilde{D}_\varepsilon(k) = -2ika_\varepsilon^+$.

To investigate the behaviour of the coefficients a_ε^+ and b_ε^+ , we have to consider the resonant and non-resonant cases separately.

4.2. Scattering coefficients in the non-resonant case. First we note that, as $\varepsilon \rightarrow 0$,

$$\tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ f_+(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(-\xi_\varepsilon, \varepsilon k) = O(\xi_\varepsilon),$$

and that, in view of (2.9),

$$\varepsilon \tilde{f}'_+(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ f'_+(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f'_-(-\xi_\varepsilon, \varepsilon k) \rightarrow -D(0).$$

It therefore follows that

$$\varepsilon a_\varepsilon^+ \rightarrow -D(0), \quad \varepsilon b_\varepsilon^+ \rightarrow D(0)$$

as $\varepsilon \rightarrow 0$. The above analysis remains in force for real k ; therefore, one gets the following result.

Corollary 4.2. *Assume that the potential V is non-resonant. Then, for every nonzero k with $\text{Im } k \geq 0$, the reflection $\tilde{r}_\varepsilon(k) := b_\varepsilon^+/a_\varepsilon^+$ and transmission $\tilde{t}_\varepsilon(k) := 1/a_\varepsilon^+$ coefficients of the Schrödinger operator \tilde{S}_ε satisfy the asymptotic relations $\tilde{r}_\varepsilon(k) \rightarrow -1$ and $\tilde{t}_\varepsilon(k) \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

This can be compared with the analogous result in [23, 24, 39] proved for the operator family S_ε in the case where the support of V is contained in $[-1, 1]$.

4.3. Scattering coefficients in the resonant case. As $\varepsilon \rightarrow 0$, we have

$$\tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ f_+(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(-\xi_\varepsilon, \varepsilon k) \rightarrow \theta^{-1},$$

by (2.6) and (2.15), while (2.8) and (2.16) yield

$$\tilde{f}'_+(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ \varepsilon^{-1} f'_+(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- \varepsilon^{-1} f'_-(-\xi_\varepsilon, \varepsilon k) \rightarrow ik\theta.$$

It therefore follows that, as $\varepsilon \rightarrow 0$,

$$a_\varepsilon^+ \rightarrow \frac{1}{2}[\theta^{-1} + \theta], \quad b_\varepsilon^+ \rightarrow \frac{1}{2}[\theta^{-1} - \theta].$$

The above analysis remains in force for real k ; therefore, one arrives at the following result.

Corollary 4.3. *Assume that the potential V is resonant and that θ is given by (1.5) (or, equivalently, by (2.11)). Then, for every nonzero k with $\text{Re } k \geq 0$, the reflection $\tilde{r}_\varepsilon(k) := b_\varepsilon^+/a_\varepsilon^+$ and transmission $\tilde{t}_\varepsilon(k) := 1/a_\varepsilon^+$ coefficients of the Schrödinger operator \tilde{S}_ε satisfy the asymptotic relations*

$$\tilde{r}_\varepsilon(k) \rightarrow \frac{1 - \theta^2}{1 + \theta^2}, \quad \tilde{t}_\varepsilon(k) \rightarrow \frac{2\theta}{1 + \theta^2} \quad \text{as } \varepsilon \rightarrow 0.$$

We note that the above limits are the reflection and transmission coefficients of the Schrödinger operator $S(\theta)$ and that they coincide with the value at $k = 0$ of the reflection and transmission coefficients for the Schrödinger operator S_1 , see [32]. This can also be compared with the analogous result in [23, 24, 39] proved for the operator family S_ε in the case where the support of V is contained in $[-1, 1]$.

4.4. Some useful estimates. We conclude this section with establishing several estimates that will essentially be used in the next section.

Lemma 4.4. *There are constants K_1 and K_2 such that the following holds:*

(i) *for all sufficiently small ε and all x with $|x| > x_\varepsilon$,*

$$|\widetilde{D}_\varepsilon^{-1}(k)\widetilde{f}_\pm(x, \varepsilon, k)| \leq K_1|e^{\pm ikx}|;$$

(ii) *for all sufficiently small ε and all $x \in \mathbb{R}$,*

$$\begin{aligned} \int_x^\infty \frac{|\widetilde{f}_+(t, \varepsilon, k)|^2}{|\widetilde{D}_\varepsilon(k)|^2} dt &\leq K_2|e^{2ikx}|, \\ \int_{-\infty}^x \frac{|\widetilde{f}_-(t, \varepsilon, k)|^2}{|\widetilde{D}_\varepsilon(k)|^2} dt &\leq K_2|e^{-2ikx}|. \end{aligned}$$

Proof. We only give details for the right Jost solution $\widetilde{f}_+(\cdot, \varepsilon, k)$, the case of $\widetilde{f}_-(\cdot, \varepsilon, k)$ being completely analogous. Item (i) follows from the fact that $\widetilde{f}(x, \varepsilon, k) = e^{ikx}$ for $x > x_\varepsilon$ and that for $x < -x_\varepsilon$

$$\frac{\widetilde{f}(x, \varepsilon, k)}{\widetilde{D}_\varepsilon(k)} = -\frac{1}{2ik} e^{ikx} - \frac{\widetilde{r}_\varepsilon(k)}{2ik} e^{-ikx},$$

where $\widetilde{r}_\varepsilon(k)$ has a finite limit as $\varepsilon \rightarrow 0$ by Corollary 4.2 in the non-resonant case and by Corollary 4.3 in the resonant one.

To prove (ii), we first estimate the integral over $(-x_\varepsilon, x_\varepsilon)$. Denote by P_ε the operator of multiplication by the function χ_ε ; then P_ε is an orthogonal projector and $\|P_\varepsilon h\| \rightarrow 0$ as $\varepsilon \rightarrow 0$ for every $h \in L_2(\mathbb{R})$. In view of Theorem 3.2, we find that

$$P_\varepsilon(\widetilde{S}_\varepsilon - k^2)^{-1}h = P_\varepsilon \left[(\widetilde{S}_\varepsilon - k^2)^{-1} - (S_0 - k^2)^{-1} \right] h + P_\varepsilon(S_0 - k^2)^{-1}h \rightarrow 0$$

in $L_2(\mathbb{R})$ as $\varepsilon \rightarrow 0$. Take now h to be the characteristic function of the interval $[-3, -1]$; then for all ε such that $x_\varepsilon < 1$ we calculate

$$\begin{aligned} P_\varepsilon(\widetilde{S}_\varepsilon - k^2)^{-1}h(x) &= \frac{\chi_\varepsilon(x)\widetilde{f}_+(x, \varepsilon, k)}{\widetilde{D}_\varepsilon(k)} \int_{-3}^{-1} e^{-ikt} dt \\ &= \frac{2e^{ik} \sin k}{k} \frac{\chi_\varepsilon(x)\widetilde{f}_+(x, \varepsilon, k)}{\widetilde{D}_\varepsilon(k)}. \end{aligned}$$

It thus follows that

$$\|P_\varepsilon(\widetilde{S}_\varepsilon - k^2)^{-1}h\|^2 = \frac{4|e^{ik} \sin k|^2}{|k|^2} \int_{-x_\varepsilon}^{x_\varepsilon} \frac{|f_+(x, \varepsilon, k)|^2}{|\widetilde{D}_\varepsilon(k)|^2} dx \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

Returning now to part (ii) of the lemma, we choose ε_0 so small that $|e^{2ikx}| > \frac{1}{2}$ for all $x \leq x_{\varepsilon_0}$. If $x > x_{\varepsilon_0}$, the desired inequality holds by (i) for all $\varepsilon < \varepsilon_0$ with $K_2 = K_1^2/(2|\operatorname{Im} k|)$; otherwise we use (i) and the above limit to get the estimate

$$\begin{aligned} \int_x^\infty \frac{|\widetilde{f}_+(t, \varepsilon, k)|^2}{|\widetilde{D}_\varepsilon(k)|^2} dt &\leq K_1^2 \int_x^\infty |e^{2ikt}| dt + \int_{-x_\varepsilon}^{x_\varepsilon} \frac{|f_+(t, \varepsilon, k)|^2}{|\widetilde{D}_\varepsilon(k)|^2} dt \\ &\leq K_2|e^{2ikx}| \end{aligned}$$

with $K_2 := 1 + K_1^2/(2|\operatorname{Im} k|)$ holding for all sufficiently small ε . \square

5. PROXIMITY OF THE OPERATOR FAMILIES S_ε AND \tilde{S}_ε

In this section we shall establish proximity of the operator families S_ε and \tilde{S}_ε in the norm resolvent sense, i.e., we shall prove the following theorem.

Theorem 5.1. *For every $k^2 \in \mathbb{C} \setminus \mathbb{R}$ it holds that*

$$\|(S_\varepsilon - k^2)^{-1} - (\tilde{S}_\varepsilon - k^2)^{-1}\| \rightarrow 0 \quad (5.1)$$

as $\varepsilon \rightarrow 0$.

Clearly, combination of Theorems 3.2 and 5.1 gives a complete proof of the claimed convergence of S_ε to S_0 .

Set

$$u_\varepsilon(x) := \varepsilon^{-1} |V(x/\varepsilon)|^{1/2} [1 - \chi_\varepsilon(x)]$$

and

$$w_\varepsilon(x) := \text{sign}(V(x/\varepsilon)) u_\varepsilon(x)$$

and denote by U_ε and W_ε the operators of multiplications by u_ε and w_ε respectively. The operators U_ε and W_ε are in general unbounded; note, however, that the L_2 -norm of the functions u_ε and w_ε vanishes as $\varepsilon \rightarrow 0$. Indeed, in view of Lemma 2.2 we deduce

$$\begin{aligned} \|u_\varepsilon\|^2 &= \|w_\varepsilon\|^2 = \frac{1}{\varepsilon^2} \int_{|t| > x_\varepsilon} \left| V\left(\frac{t}{\varepsilon}\right) \right| dt = \frac{1}{\varepsilon} \int_{|s| > \xi_\varepsilon} |V(s)| ds \\ &\leq \frac{1}{\varepsilon} \int_{|s| > \xi_\varepsilon} \frac{\rho_V(s)}{\rho_V(\xi_\varepsilon)} |V(s)| ds = \int_{|s| > \xi_\varepsilon} \rho_V(s) |V(s)| ds = o(1) \end{aligned} \quad (5.2)$$

as $\varepsilon \rightarrow 0$.

The relation $S_\varepsilon = \tilde{S}_\varepsilon + U_\varepsilon W_\varepsilon$ yields the following well-known (formal) representation of the resolvent $R_\varepsilon(k) := (S_\varepsilon - k^2)^{-1}$ of S_ε in terms of the resolvent $\tilde{R}_\varepsilon(k) := (\tilde{S}_\varepsilon - k^2)^{-1}$ of \tilde{S}_ε and the perturbation:

$$R_\varepsilon(k) - \tilde{R}_\varepsilon(k) = \tilde{R}_\varepsilon(k) W_\varepsilon [I + U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon]^{-1} U_\varepsilon \tilde{R}_\varepsilon(k). \quad (5.3)$$

We shall prove that the norms of the operators $\tilde{R}_\varepsilon(k) W_\varepsilon$, $U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon$, and $U_\varepsilon \tilde{R}_\varepsilon(k)$ vanish as $\varepsilon \rightarrow 0$; this will justify both formula (5.3) and the claim of Theorem 5.1.

5.1. The norm of special integral operators. It is immediate to see that the operators $\tilde{R}_\varepsilon(k) W_\varepsilon$, $U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon$, and $U_\varepsilon \tilde{R}_\varepsilon(k)$ are integral ones and act on a function $y \in L_2(\mathbb{R})$ via

$$\frac{\varphi_+(x)}{\tilde{D}_\varepsilon(k)} \int_{-\infty}^x \varphi_-(t) y(t) dt + \frac{\varphi_-(x)}{\tilde{D}_\varepsilon(k)} \int_x^\infty \varphi_+(t) y(t) dt,$$

where φ_+ equals one of the functions $\tilde{f}_+(\cdot, \varepsilon, k)$, $\tilde{f}_+(\cdot, \varepsilon, k) w_\varepsilon$ or $\tilde{f}_+(\cdot, \varepsilon, k) u_\varepsilon$, and similarly for φ_- . To estimate the norm of such operators, we use the following result (see, e.g., [37]):

Proposition 5.2. *Assume that ψ_- and ψ_+ are functions belonging to $L^2_{2,\text{loc}}(\mathbb{R})$ and let T_\pm be integral operators defined via*

$$T_- y(x) = \psi_+(x) \int_{-\infty}^x \psi_-(t) y(t) dt$$

and

$$T_+ y(x) = \psi_-(x) \int_x^\infty \psi_+(t) y(t) dt$$

respectively. Then T_{\pm} are bounded in $L_2(\mathbb{R})$ if and only if the quantity

$$K := \sup_{x \in \mathbb{R}} \left(\int_{-\infty}^x |\psi_{-}(t)|^2 dt \cdot \int_x^{\infty} |\psi_{+}(t)|^2 dt \right)^{1/2}$$

is finite; in this case $\|T_{\pm}\| \leq 2K$.

5.2. The operator $U_{\varepsilon} \tilde{R}_{\varepsilon}(k) W_{\varepsilon}$. This is an integral operator of the form

$$\begin{aligned} U_{\varepsilon} \tilde{R}_{\varepsilon}(k) W_{\varepsilon} y(x) &= \frac{u_{\varepsilon}(x) \tilde{f}_{+}(x, \varepsilon, k)}{\tilde{D}_{\varepsilon}(k)} \int_{-\infty}^x y(t) w_{\varepsilon}(t) \tilde{f}_{-}(t, \varepsilon, k) dt \\ &\quad + \frac{u_{\varepsilon}(x) \tilde{f}_{-}(x, \varepsilon, k)}{\tilde{D}_{\varepsilon}(k)} \int_x^{\infty} y(t) w_{\varepsilon}(x) \tilde{f}_{+}(t, \varepsilon, k) dt. \end{aligned}$$

By virtue of Proposition 5.2 and the relation $|u_{\varepsilon}| = |w_{\varepsilon}|$ we conclude that the norm of $U_{\varepsilon} \tilde{R}_{\varepsilon}(k) W_{\varepsilon}$ vanishes as $\varepsilon \rightarrow 0$ provided this is true for the quantity $\sup_{x \in \mathbb{R}} K_{\varepsilon}(x)$, with

$$K_{\varepsilon}(x) := \frac{1}{|\tilde{D}_{\varepsilon}(k)|^2} \int_{-\infty}^x |u_{\varepsilon}(t) \tilde{f}_{-}(t, \varepsilon, k)|^2 dt \cdot \int_x^{\infty} |u_{\varepsilon}(t) \tilde{f}_{+}(t, \varepsilon, k)|^2 dt.$$

For $x \geq x_{\varepsilon}$ we have

$$\int_x^{\infty} |u_{\varepsilon}(t) \tilde{f}_{+}(t, \varepsilon, k)|^2 dt = \int_x^{\infty} |u_{\varepsilon}(t) e^{ikt}|^2 dt \leq |e^{2ikx}| \|u_{\varepsilon}\|^2;$$

similarly, by Lemma 4.4(i)

$$\frac{1}{|\tilde{D}_{\varepsilon}(k)|^2} \int_{-\infty}^x |u_{\varepsilon}(t) \tilde{f}_{-}(t, \varepsilon, k)|^2 dt \leq K_1^2 |e^{-2ikx}| \|u_{\varepsilon}\|^2. \quad (5.4)$$

Therefore,

$$\sup_{x \geq x_{\varepsilon}} K_{\varepsilon}(x) \leq K_1^2 \|u_{\varepsilon}\|^4 \rightarrow 0$$

as $\varepsilon \rightarrow 0$ due to (5.2). For $x \leq -x_{\varepsilon}$ the arguments exploit the relation $\tilde{f}_{-}(x, \varepsilon, k) = e^{-ikx}$ and the estimate of Lemma 4.4(i) to conclude that

$$\sup_{x \leq -x_{\varepsilon}} K_{\varepsilon}(x) \leq K_1^2 \|u_{\varepsilon}\|^4 \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Finally,

$$\sup_{|x| < x_{\varepsilon}} K_{\varepsilon}(x) \leq \frac{|e^{4ikx_{\varepsilon}}| \|u_{\varepsilon}\|^4}{|\tilde{D}_{\varepsilon}(k)|^2} \rightarrow 0$$

as $\varepsilon \rightarrow 0$, thus proving the following statement.

Lemma 5.3. *The norm of the operator $U_{\varepsilon} \tilde{R}_{\varepsilon}(k) W_{\varepsilon}$ vanishes as $\varepsilon \rightarrow 0$.*

5.3. The operators $\tilde{R}_{\varepsilon}(k) W_{\varepsilon}$ and $U_{\varepsilon} \tilde{R}_{\varepsilon}(k)$. We need to show that both $\sup_{x \in \mathbb{R}} K_{\varepsilon}^{(1)}(x)$ and $\sup_{x \in \mathbb{R}} K_{\varepsilon}^{(2)}(x)$ vanish as $\varepsilon \rightarrow 0$, with

$$K_{\varepsilon}^{(1)}(x) := \frac{1}{\tilde{D}_{\varepsilon}^2(k)} \int_{-\infty}^x |u_{\varepsilon}(t) \tilde{f}_{-}(t, \varepsilon, k)|^2 dt \cdot \int_x^{\infty} |\tilde{f}_{+}(t, \varepsilon, k)|^2 dt$$

and

$$K_{\varepsilon}^{(2)}(x) := \frac{1}{\tilde{D}_{\varepsilon}^2(k)} \int_{-\infty}^x |\tilde{f}_{-}(t, \varepsilon, k)|^2 dt \cdot \int_x^{\infty} |u_{\varepsilon}(t) \tilde{f}_{+}(t, \varepsilon, k)|^2 dt.$$

Since both quantities can be estimated in a similar way, only the first one will be treated in detail.

For $x \geq x_\varepsilon$ we have

$$\int_x^\infty |\tilde{f}_+(t, \varepsilon, k)|^2 dt = \frac{1}{2|\operatorname{Im} k|} |e^{2ikx}|,$$

which together with (5.4) results in

$$\sup_{x \geq x_\varepsilon} K_\varepsilon^{(1)}(x) \leq \frac{K_1^2}{2|\operatorname{Im} k|} \|u_\varepsilon\|^2.$$

For $x < x_\varepsilon$ we use Lemma 4.4(ii) and the inequality

$$\int_{-\infty}^x |u_\varepsilon(t) \tilde{f}_-(t, \varepsilon, k)|^2 dt \leq |e^{-2ikx}| \|u_\varepsilon\|^2$$

to conclude that

$$\sup_{x < x_\varepsilon} K_\varepsilon^{(1)}(x) \leq K_2 \|u_\varepsilon\|^2.$$

Combining the above estimates with (5.2), we arrive at the following conclusion.

Lemma 5.4. *Under the standing assumptions,*

$$\lim_{\varepsilon \rightarrow 0} \left(\|U_\varepsilon \tilde{R}_\varepsilon(k)\| + \|\tilde{R}_\varepsilon(k) W_\varepsilon(k)\| \right) = 0.$$

Proof of Theorem 5.1. It suffices to note that Lemmata 5.3 and 5.4 justify equation (5.3) and, in turn, establish the convergence in (5.1). \square

The main result of the paper is now easy to justify:

Proof of Theorem 1.1. Clearly, the statements of Theorems 3.2 and 5.1 immediately yield the norm resolvent convergence, as $\varepsilon \rightarrow 0$, of the family of Schrödinger operators S_ε given by (1.1) to the limiting operator S_0 . \square

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